Backlund transformations and explicit solutions of certain inhomogeneous nonlinear Schrodinger-type equations

1991 J. Phys. A: Math. Gen. 24 L337

(http://iopscience.iop.org/0305-4470/24/7/005)

J. Phys. A: Math. Gen. 24 (1991) L337–L343. Printed in the UK

LETTER TO THE EDITOR

Bäcklund transformations and explicit solutions of certain inhomogeneous nonlinear Schrödinger-type equations

K Porsezian

Centre for Nonlinear Dynamics, Department of Physics, Bharathidasan University, Tiruchirapalli 620 024, India

Received 28 November 1990

Abstract. We show that the Darboux–Bargmann method can be used to derive Bäcklund transformations and construct explicit soliton solutions for certain inhomogeneous nonlinear Schrödinger-type equations.

Nonlinear partial differential equations integrable by the inverse spectral transform (IST) method form a wide class of soliton equations which possess many remarkable properties [1–3]. These soliton solutions can be obtained in different ways.

Among them, the Bäcklund transformation (BT) technique is one of the direct methods to obtain the soliton solutions of a given NLEE [4–6] and has the additional merit of providing the wavefunctions associated with the linear eigenvalue problem. Using the Darboux–Bargmann method, such an approach has been successfully applied to certain NLEE [7, 8]. The main aim of this letter is to show that the above method can also be used to derive the BT and soliton solutions for a class of inhomogeneous nonlinear Schrödinger (NLS) type equations.

Consider the ZS/AKNS eigenvalue problem defined in the form

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix}_x = \begin{pmatrix} -\mathrm{i}\lambda & q \\ r & \mathrm{i}\lambda \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \tag{1}$$

where λ is an eigenvalue parameter which is non-isospectral depending on the inhomogeneities involved in the system. The corresponding time evolution equation is given by

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix}_t = \begin{pmatrix} A & B \\ C & -A \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \tag{2}$$

where A, B and C are functions of λ, q and r. From the compatibility condition $V_{xt} = V_{tx}$, we obtain various nonlinear evolution equations of physical interest depending on the choices of A, B, C and r. For nonlinear Schrödinger (NLS) type equations $r = -q^*$.

To construct the BT of (1) and (2) with $r = -q^*$, we define a new set of linear evolution equations for $V'_{1,2}$ similar to (1) and (2) with all quantities replaced by the primed ones except the eigenvalue λ which remains the same

$$\begin{pmatrix} V'_1 \\ V'_2 \end{pmatrix}_x = \begin{pmatrix} -\mathrm{i}\lambda & q' \\ -q'^* & \mathrm{i}\lambda \end{pmatrix} \begin{pmatrix} V'_1 \\ V'_2 \end{pmatrix} \tag{3}$$

0305-4470/91/070337+07$03.50

and

$$\begin{pmatrix} V_1' \\ V_2' \end{pmatrix}_t = \begin{pmatrix} A' & B' \\ C' & -A' \end{pmatrix} \begin{pmatrix} V_1' \\ V_2' \end{pmatrix} \tag{4}$$

where A', B' and C' are functions of λ, q' and q'^*.

Now, the quantities referring to the n-soliton problem are defined by

$$V_1' \equiv V_1(n) \qquad V_2' \equiv V_2(n) \qquad q' \equiv q_n \qquad q'^* \equiv q_n^* \qquad V' \equiv \begin{pmatrix} V_1' \\ V_2' \end{pmatrix}. \tag{5}$$

Similarly, those corresponding to the $(n-1)$ soliton problem are given by

$$\begin{aligned} &V_1 \equiv V_1(n-1) \qquad V_2 \equiv V_2(n-1) \qquad q \equiv q(n-1) \\ &q^* \equiv q^*(n-1) \qquad V \equiv \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}. \end{aligned} \tag{6}$$

We turn to Darboux's method [9] to expand the n-soliton wavefunction V' in terms of $(n-1)$-soliton solutions V:

$$\begin{pmatrix} V_1' \\ V_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \tag{7}$$

where a, b, c and d are functions of (q, q^*) and (q', q'^*) and through them functions of x and t. Substituting (7) in (1)-(4) and equating the coefficients of $V_{1,2}$ on both sides we obtain the following differential equations:

$$\begin{aligned} a_x &= q^* b + q' c \\ b_x &= -2\mathrm{i}\lambda b - qa + q'd \\ c_x &= 2\mathrm{i}\lambda c - q'^* A + q^* d \\ d_x &= -q'^* b - qc. \end{aligned} \tag{8}$$

We now invoke an idea due to Bargmann [10] who showed that for the Schrödinger equation $-(\mathrm{d}^2\psi/\mathrm{d}x^2) + V(x)\psi = k^2\psi$, and for a potential capable of giving n bound states, the wavefunction can be written in the form $\mathrm{e}^{\mathrm{i}kx} f(k, x)$, where f is an nth-degree polynomial in k. It is well known that the Schrödinger equation is the associated eigenvalue equation for the application of the inverse scattering transform method to the KdV equation [11]. These considerations have been extended to the entire AKNS problem [12]. On the other hand, we know that the n-soliton solution of the NLEE, envisaged by the ZS/AKNS equation can be looked upon as a potential giving n bound states. Thus the idea due to Bargmann suggests that V and V' will differ by a linear function of λ since they satisfy the eigenvalue equation involving the first power of λ. Thus we write

$$a = a_0 + \lambda a_1 \qquad b = b_0 + \lambda b_1 \qquad c = c_0 + \lambda c_1 \qquad d = d_0 + \lambda d_1 \tag{9}$$

where a, b, c, d are functions of x and t through (q, q^*) and (q', q'^*). Now using (9) in (8) and equating the coefficients of equal powers of λ on both sides, we get a set of algebraic and differential equations for $a, \ldots, d$. Solving the resultant equations, we obtain

$$a_1=\mu(t) \qquad b_1=c_1=0 \qquad d_1=\delta(t)$$

$$a_0=\mathrm{i}\alpha\pm\tfrac{1}{2}\sqrt{4\beta^2-|q'+q|^2} \qquad d_0=-\left(\frac{\delta}{\mu}\right)a_0+\beta(t) \tag{10}$$

$$b_0=\frac{\mu}{2\mathrm{i}}\left[\left(\frac{\delta}{\mu}\right)q'-q\right] \qquad c_0=\frac{\mu}{2\mathrm{i}}\left[q'^*-\left(\frac{\delta}{\mu}\right)q^*\right]$$

where μ, δ, α, β are the integration constants which may depend on time. In addition, with the proper choices of integration constants, the following equation must be satisfied

$$q'_x+q_x=-2\mathrm{i}\alpha(q'+q)+(q'-q)[4\beta^2-|q'+q|^2]^{1/2}. \tag{11}$$

Equation (11) gives the space BT for the NLS type equations obtained from the ZS/AKNS eigenvalue problem defined in (1) and (2) with $r=-q^*$. Similarly, from the time evolution part of the eigenvalue problem, we obtain the following evolution equations connecting n and $(n-1)$ soliton solutions:

$$a_t=a(A'-A)+cB'-bC \tag{12a}$$

$$b_t=b(A'+A)-aB+dB' \tag{12b}$$

$$c_t=-c(A'+A)-aC'-dC \tag{12c}$$

$$d_t=-d(A'-A)+bC'-cB. \tag{12d}$$

From (10), we see that (12b, c) would give the time part of the BTs. Below, we discuss, the BTs and the construction of soliton solutions for certain inhomogeneous nonlinear Schrödinger-type equations.

The generalized Hirota equation with linearly x-dependent inhomogeneities is given by [13]

$$\mathrm{i}q_t+\mathrm{i}\mu_1 q+\mathrm{i}(\gamma_1+\mu_1 x)q_x+(\gamma_2+\mu_2 x)(q_{xx}+2|q|^2q)$$
$$+2\mu_2\left(q_x+q\int_{-\infty}^{x}|q|^2\,\mathrm{d}x'\right)+\mathrm{i}\gamma(q_{xxx}+6|q|^2q_x)=0 \tag{13}$$

where the values of A, B and C in the eigenvalue equation (2) take the form

$$A=\mathrm{i}\mu_2\int_{-\infty}^{x}|q|^2\,\mathrm{d}x'+\mathrm{i}(\gamma_2+\mu_2 x)|q|^2+\gamma(qq_x^*-q^*q_x)-4\mathrm{i}\gamma\lambda^3$$
$$-2\mathrm{i}(\gamma_2+\mu_2 x)\lambda^2+\mathrm{i}(\gamma_1+\mu_1 x)\lambda+2\mathrm{i}\gamma\lambda|q|^2$$
$$B=\mathrm{i}(\gamma_2+\mu_2 x)q_x-\gamma q_{xx}-(\gamma_1+\mu_1 x)q+\mathrm{i}\mu_2 q-2\gamma|q|^2q+4\gamma\lambda^2q \tag{14}$$
$$+2\mathrm{i}\gamma\lambda q_x+2(\gamma_2+\mu_2 x)\lambda q$$
$$C=-B^*.$$

The compatibility condition $(V_x)_t=(V_t)_x$ leads to (13) only when λ satisfies the equation

$$\lambda_t=2\mu_2\lambda^2-\mu_1\lambda \qquad \text{with } \lambda=\alpha_1(t)+\mathrm{i}\beta_1(t) \tag{15}$$

so that the flow is non-isospectral.

Now, we will derive the soliton solutions and wavefunctions for the generalized NLSE. By introducing the transformation

$$\Gamma = \frac{V_1}{V_2}\bigg| = \alpha_1 + i\beta_1 \qquad \beta_1 > 0$$

equations (1) and (2) reduce to the Riccati equations

$$\Gamma_x = -2i\lambda\Gamma + q^*\Gamma^2 - q \tag{16}$$

$$\Gamma_t = B + 2A\Gamma - C\Gamma^2 \tag{17}$$

with Γ' satisfying similar form of equations. Choosing $\Gamma' = 1/\Gamma^*$, we obtain the following relation between n and $(n-1)$ solitons [4]

$$q' + q = -\frac{4\beta_1\Gamma}{(1+|\Gamma|^2)} \qquad \Gamma = -\frac{2\beta_1 + \sqrt{4\beta_1^2 - |q'+q|^2}}{(q'^* + q^*)}. \tag{18}$$

From (18) one can derive with a recursive method the complete soliton solutions of the generalized NLS equation and the corresponding ZS/AKNS wavefunctions. To get the soliton solution of a given nonlinear equation, we start with the zero soliton solution $q(0) = 0$ of the generalized NLS equation and using this in (1), the zero soliton ZS/AKNS wavefunction $V(0)$ is obtained. The time dependence of $V(0)$ is derived from (2) with the proper choices of A, B and C. We use these known $q(0)$ and $V(0)$ in (18) to obtain the one soliton solution of the given equation. The procedure can then be continued to obtain multi-soliton solutions and the accompanying wavefunctions. For instance, the one-soliton solution of (13) is found to be

$$q = -2\beta_1 \operatorname{sech} \xi_1 \, e^{-i\xi\chi_1} \tag{19}$$

where

$$\xi_1 = 2\beta_1 x - 2\int [\gamma_1\beta_1 - 4\gamma_2\alpha_1\beta_1 + 4\gamma\beta_1^2 - 12\gamma\alpha_1^2\beta_1]\, dt'$$

$$\chi_1 = 2\alpha_1 x + 2\int [\gamma_1\alpha_1 - \gamma_2(\alpha_1^2 - \beta_1^2) - 4\gamma(\alpha_1^3 - 3\alpha_1\beta_1^2)]\, dt'.$$

We consider the evolution equation in the form [14, 15] (by putting $N = 2$ in equation (17) of [14]):

$$iq_t + q_{rr} + 2|q|^2 q + \frac{q_r}{r} - \frac{q}{r^2} + 4q\int \frac{|q|^2}{r'}\, dr' = 0. \tag{20}$$

The linear eigenvalue problem associated with (20) is given by

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix}_r = \begin{pmatrix} -\dfrac{i\lambda r}{2} & q \\ -q^* & \dfrac{i\lambda r}{2} \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \tag{21a}$$

and

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix}_t = \begin{pmatrix} \dfrac{-i\lambda^2 r^2}{2} + 2i\left(\displaystyle\int \frac{|q|^2}{r'}\, dr' + \frac{|q|^2}{2}\right) & \lambda r q + i(q_r + q/r) \\ -\lambda r q^* + i(q_r^* + q^*/r) & \dfrac{i\lambda^2 r^2}{2} - 2i\left(\displaystyle\int \frac{|q|^2}{r'}\, dr' + \frac{|q|^2}{2}\right) \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \tag{21b}$$

and $\lambda_t = 2\lambda^2$, so that the compatibility condition leads to (20). Then the relation between the n and $(n-1)$ soliton solutions reads

$$q' + q = -\frac{4\beta_1 r \Gamma}{(1+|\Gamma|^2)} \tag{22}$$

with

$$\Gamma = -\frac{2\beta_1 r + \sqrt{4\beta_1^2 r^2 - |q'+q|^2}}{(q'^* + q^*)}. \tag{23}$$

Now, the spatial BT associated with (20) can be constructed as

$$q_r + q'_r = \left(\frac{q+q'}{r}\right) + \tfrac{1}{2}(q'-q)[4\beta_1^2 r^2 - |q'+q|^2]^{1/2} - \mathrm{i}\alpha_1 \gamma (q'+q) \tag{24}$$

and the time derivative of (22) with (21) gives the time BT. From (22) the one-soliton solution for (20) is found to be

$$q(1) = -2\beta_1 r \operatorname{sech}\left(\frac{\beta_1 r^2}{2} + \Delta_1\right) \exp\left[-\mathrm{i}\left(\frac{\alpha_1 r^2}{2} + \Delta_2\right)\right] \tag{25}$$

with

$$\alpha_1 = \frac{\alpha_1(0) + 2t}{(\alpha_1(0)+2t)^2 + \beta_1^2(0)} \qquad \beta_1 = \frac{\beta_1(0)}{(\alpha_1(0)+2t)^2 + \beta_1^2(0)}.$$

Here also we find that soliton spreads as a function of time (figure 1), i.e. the soliton represents a ring wave whose radius and thickness increases linearly with time (for large t), while the amplitude decreases like t^{-1} for fixed r.

It is interesting to note that under the transformations $x = r^2/4$, $q = 2\hat{q}/r$, we observe that for the parametric choice $\mu_2 = 1$, $\gamma_2 = \gamma_1 = \mu_1 = 0$, systems (13) and (20) are the same.

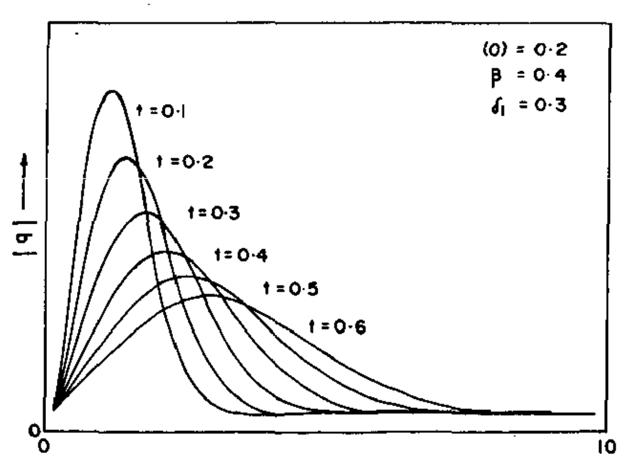

Figure 1.

In dimensionless form the NLS equation in a weakly inhomogeneous medium can be written as [16]

$$\mathrm{i}q_t + q_{xx} + 2(|q|^2 - \rho x)q = 0. \tag{26}$$

The associated eigenvalue problems read

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix}_x = \begin{pmatrix} -\mathrm{i}\lambda & q\,\mathrm{e}^{A} \\ -q^*\,\mathrm{e}^{-A} & \mathrm{i}\lambda \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \tag{27a}$$

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}_t = \begin{bmatrix} \mathrm{i}|q|^2 - 2\mathrm{i}\lambda^2 - 4\mathrm{i}\rho\lambda t & [\mathrm{i}(q_x - 2\mathrm{i}\rho t q) + 2\lambda q]\,\mathrm{e}^{A} \\ [\mathrm{i}(q_x^* + 2\mathrm{i}\rho t q^*) - 2\lambda q^*) - 2\lambda q^*]\,\mathrm{e}^{-A} & -\mathrm{i}|q|^2 + 2\mathrm{i}\lambda^2 + 4\mathrm{i}\rho\lambda t \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \tag{27b}$$

where $A = 2\mathrm{i}\rho x t + \frac{4}{3}\mathrm{i}\rho^2 t^3$. Following the same method discussed earlier and introducing a function $\Gamma = V_1/V_2$, and choosing $\Gamma' = 1/\Gamma^*$, the relation between n and $(n-1)$ soliton solutions is found to be

$$q' + q = \frac{-4\beta_1 \Gamma\,\mathrm{e}^{-A}}{(1 + |\Gamma|^2)}. \tag{28}$$

Then, the BTs associated with (26) can be written as

$$q_x + q'_x = \{(q - q')[4\beta_1^2 - |q' + q|^2]^{1/2} - 2\mathrm{i}\alpha_1(q + q')\}\,\mathrm{e}^{-A} - 2\mathrm{i}\rho t(q' + q)\,\mathrm{e}^{A}. \tag{29}$$

Similarly, by taking the time derivative of (28) and substituting (27*b*) one can easily obtain the time BT. From (28), the one-soliton solution for (26) is obtained in the form

$$q(1) = -2\beta_1\,\mathrm{sech}\{2\beta_1 x + \beta_1\Delta_1 + 8\alpha_1\beta_1 t + 2\rho\beta_1 t^2\}\}$$
$$\times \exp\{-2\mathrm{i}[\alpha_1 x + 2(\alpha_1^2 - \beta_1^2)t + 2\rho\alpha_1 t^2 + \rho x t + \tfrac{2}{3}\rho^2 t^3 + \alpha_1\Delta_2]\} \tag{30}$$

which is in agreement with the solution given by Chen and Lin [16].

Thus, we conclude that the Darboux–Bargmann method can also be successfully used to study the inhomogeneous equations and we established this by working out the BTs, soliton solutions and wavefunctions for the generalized Hirota and circularly symmetric NLS equations.

The author is thankful to Professor M Lakshmanan for his guidance and critical comments on this work. He also wishes to thank Dr M Daniel for his useful discussions. He wishes to thank the Council of Scientific and Industrial Research, Government of India for the financial support in the form of a Senior Research Fellowship.

References

[1] Zakharov V E and Shabat A B 1972 *Sov. Phys.-JETP* **34** 62
[2] Ablowitz M J and Segur H 1981 *The Solitons and the Inverse Scattering Transformations* (Philadelphia, PA: SIAM)
[3] Wahlquist H D and Estabrook F B 1973 *Phys. Rev. Lett.* **31** 1386
[4] Wadati M, Sunuki H and Konno K 1975 *Prog. Theor. Phys.* **53** 419
[5] Lamb G L Jr 1974 *J. Math. Phys.* **15** 2157
[6] Levi O, Rognisco D and Sym A 1982 *Lett. Nuovo Cimento* **33** 401
[7] Rangwala A A and Rao J A 1985 *Phys. Lett.* **112A** 188
[8] Rao J A and Rangawala A A 1988 *Solitons: Introduction and Applications* ed M Lakshmanan (Berlin: Springer)

[9] Darboux G 1915 *Leçons sur la Théorie Générale des Surfaces et les Applications Géometriques du Calcul Infinitésimal* vol 2 (Paris: Gauthier-Villars) p 210
[10] Bargmann V 1949 *Rev. Mod. Phys.* **21** 488
[11] Gardner C S, Greene J M, Kruskal M D and Miura R M 1974 *Commun. Pure. Appl. Math.* **27** 97
[12] Ablowitz M J, Kaup D J, Newell A C and Segur H 1974 *Stud. Appl. Math.* **53** 249
[13] Lakshmanan M and Ganesan S 1985 *Physica* **132A** 117
[14] Lakshmanan M and Daniel M 1981 *Physica* **107A** 533
[15] Lakshmanan M and Porsezian K 1990 *Phys. Lett.* **146A** 329
[16] Chen H H and Liu C S 1978 *Phys. Fluids* **21** 377